Qu1

Sunday, December 6, 2015 10:56 AM

Let $H_b = \{x+ib : x \in \mathbb{R}\}$ where $b \in \mathbb{R}$

The aim is find $f(H_b)$.

① Let $f(x+ib) = u+iv$ and calculate that

$$u = x^2 - x - b^2, \quad v = b(2x-1)$$

② Special case that $b=0$

Show that $f(H_0) = [-\frac{1}{4}, \infty) \subset \mathbb{R}$

③ General case that $b \neq 0$

Eliminate x to get the equation

$$u = \frac{1}{4}\frac{v^2}{b^2} - \left(\frac{1}{4} + b^2\right)$$

This defines a parabola, called P

Argue that $f(H_b) \subset P$ and

$f(H_b) \supset P$

Thus, $f(H_b) = P$.

④ Find $f(V_a)$, $V_a = \{a+iy : y \in \mathbb{R}\}$, $a \in \mathbb{R}$

The method is similar.

Qu2

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(a) Just write down

$$\sigma(z) = \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

(b) A great circle on S^2 can be obtained by cutting the sphere by a plane passing through the origin, i.e.,

$$a x_1 + b x_2 + c x_3 = 0 \quad \text{for } (x_1, x_2, x_3) \in S^2$$

The great circle will not pass through the north pole $(0,0,1)$ if $c \neq 0$

In that case, we may assume

$$a x_1 + b x_2 + x_3 = 0$$

Now, the question is indeed proving:

Let $S = \{ z \in \mathbb{C} : |z - z_0| = R \}$

$\sigma(S)$ is a great circle, i.e., $\exists\, a, b \in \mathbb{R}$

$$a \cdot \frac{2\operatorname{Re} z}{|z|^2+1} + b \cdot \frac{2\operatorname{Im} z}{|z|^2+1} + \frac{|z|^2-1}{|z|^2+1} = 0$$

$$\Leftrightarrow \quad |z_0|^2 + 1 = R^2$$

Just compare the equations

$$|z - z_0|^2 = R^2 \quad \text{and}$$

$$2a \operatorname{Re}(z) + 2b \operatorname{Im}(z) + |z|^2 - 1 = 0$$

and see the equivalence

Qu3

Sunday, December 6, 2015 11:12 AM

Simply calculate u_x, u_y, v_x, v_y
for $u = 2xy + 3x^2$ and $v = -2xy^2$

Solve for (x,y) from the
Cauchy-Riemann Equations and have
$(x,y) = (0,0)$ or $\left(\frac{1}{25}, \frac{1}{5}\right)$

Does the set $\left\{(0,0), \left(\frac{1}{25}, \frac{1}{5}\right)\right\}$ contain an
open ball ?

From the above, conclude that
f is nowhere analytic

Qu4

Sunday, December 6, 2015 9:16 PM

(a) First, need to prove $f(\mathbb{D}) \subset \mathbb{D}$ by

$$|f(z)| < 1 \quad \text{if} \quad |z| < 1$$

This can be done by calculating

$$f(z) \cdot \overline{f(z)} - 1 < 0$$

Second, in $w = f(z)$, rewrite

z in terms of w for $|w| < 1$

This shows that f is onto and

it is also one-to-one.

(b) Simply use the quotient rule and

calculation in part (a).

Qu5

Sunday, December 6, 2015 9:25 PM

(a) Very straight forward method of integrating the Cauchy-Riemann Equations one by one, finally get,

$$v(x,y) = 2\tan^{-1}\frac{y}{x} + \underset{v(x_0,y_0)}{\underset{\parallel}{\text{constant}}}$$

(b) The above calculation requires that $x \neq 0$ at any moment.

Thus, $\Omega \not\supset$ y-axis.

Ω = Left half plane or right half which should be consistent with the choice of (x_0, y_0)

No matter Ω = Right or Left half plane

$$u(x,y) = \log(x^2+y^2)$$
$$v(x,y) = \tan^{-1}\frac{y}{x} + v(x_0,y_0)$$

are of C^1 on Ω, and satisfy Cauchy-Riemann Equations by the above.

$\therefore$ $u+iv$ is analytic on Ω.

Qu6

Sunday, December 6, 2015 10:31 PM

(a) In terms of rectangular coordinates

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Use Chain Rule to obtain u_x, v_x or u_y, v_y in terms of u_r, u_θ or v_r, v_θ

It leads to $f'(z) = \frac{1}{r}(\cos\theta - i\sin\theta)\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$

$$= (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

(b) One may further use chain Rule on

$$u = \rho\cos\varphi, \quad v = \rho\sin\varphi$$

Faster Method: Consider

$$\log f(z) = \log(u+iv) = \log \rho e^{i\varphi}$$
$$= \log\rho + i\varphi$$

Apply Polar Form Cauchy-Riemann Equations to the function $\log f(z)$

The final equations are

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = \frac{1}{r}\frac{\partial \varphi}{\partial \theta}, \quad \frac{-1}{r\rho}\frac{\partial \rho}{\partial \theta} = \frac{\partial \varphi}{\partial r}$$